Created in COMSOL Multiphysics 6.2

Time-Dependent Optimization

Introduction

Nonlinear systems that are driven by a sinusoidal excitation often evolve toward a periodic steady-state solution. Such systems occur in electromagnetics, plasma physics, and electrochemistry.

Model Definition

The test model solves the following ordinary differential equation:

$$\left(\frac{1}{f}\right)\frac{du}{dt} = a\sin^2(\omega t) - bu - cu^2 \tag{1}$$

where a is 0.25, b is 0.05, and c is 0.015. The frequency, f, is set to 1. Because the equation is nonlinear (due to the u^2 term), it cannot be reformulated in the frequency domain by taking its Fourier transform. This ordinary differential equation is representative of the evolution of electronically excited metastable states in a capacitively coupled plasma. The initial value of u is set to be 0.25. For these conditions, it takes about 100 periods before u reaches its periodic steady-state solution. In a real plasma, it can take more than $100{,}000$ RF cycles before the plasma has attained its periodic steady-state solution. Solving such a problem for so many RF cycles creates an insurmountable computational burden.

The periodic steady-state solution can be immediately computed using time-dependent optimization. A control variable, u_0, is used as the initial condition for Equation 1. Next, an objective function is defined as:

$$g = (u - u_0)^2$$

When performing time-dependent optimization, the objective function is only evaluated at the last solution time. Thus, the global objective function seeks to make the initial value of u equal to the final value of u after exactly one period. This corresponds to the periodic steady-state solution to the problem.

Results and Discussion

The time evolution of u is plotted in Figure 1. A close-up of the final few periods is plotted in Figure 2. This shows that u has indeed reached its periodic steady-state solution.

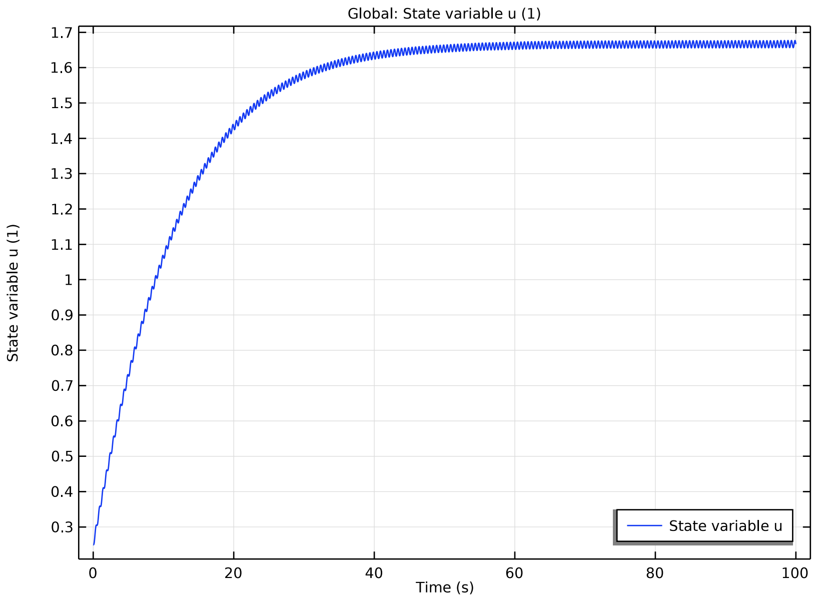

Figure 1: Plot of the evolution of u from its initial value of 0.25. There is a slow, steady increase in u over the first several periods along with oscillations at twice the driving frequency.

It is also obvious from Figure 2 that over 1 period, the value of u at the beginning of the period is the same as at the end of the period. In Figure 3 the solution computed by the optimization solver is shown. Note that the forward problem is only solved for 1 period. In total the optimization solver computes the solution to the forward problem only 6 times resulting in a much reduced simulation time.

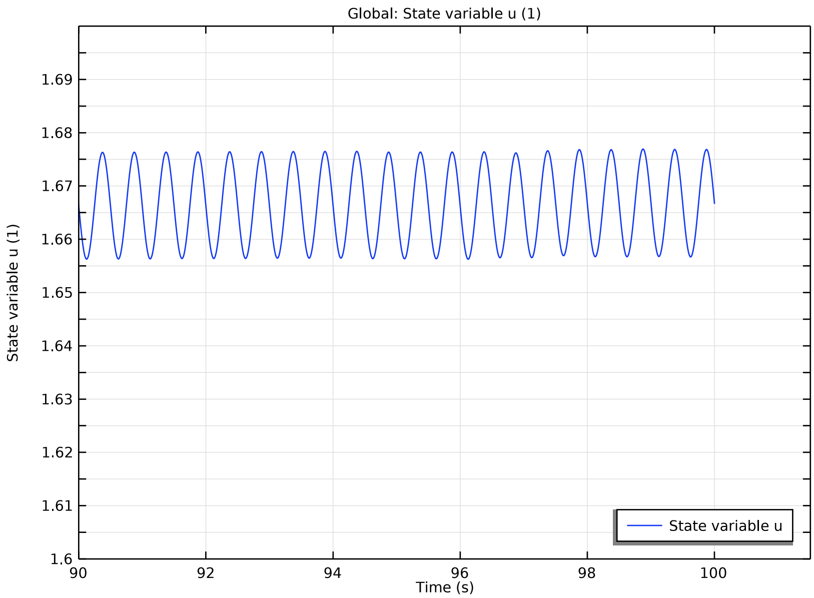

Figure 2: Close up of the last several cycles of the forward problem. The model has clearly reached its periodic steady state solution after 100 cycles.

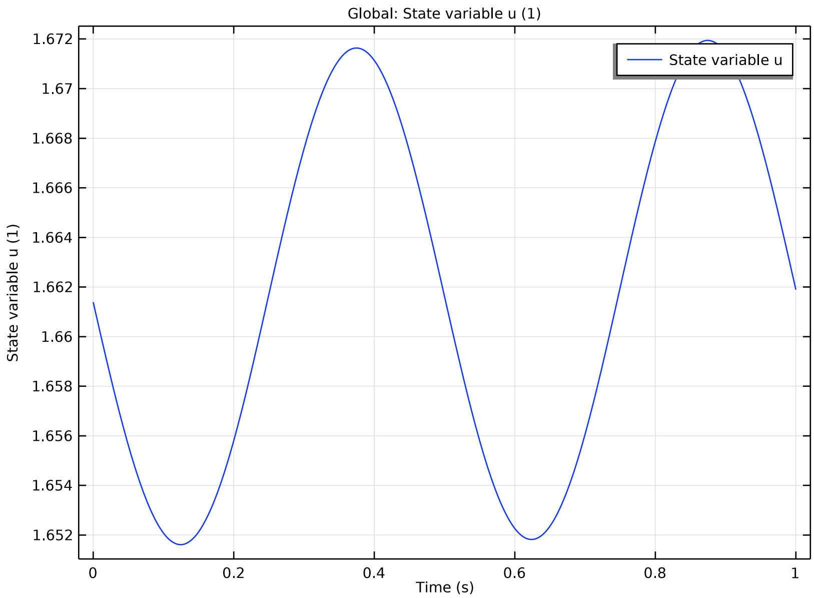

Figure 3: Plot of the solution computed by the optimization solver.

Reference

1. D.P. Lymberopoulos and D.J. Economou, “Fluid simulation of glow discharges: Effect of metastable atoms in argon,” *J. Appl. Phys.* vol. 73, no. 8, 1993.

Application Library path: `Optimization_Module/Parameter_Estimation/time_dependent_optimization`

Modeling Instructions

From the **File** menu, choose **New**.

NEW

In the **New** window, click **Model Wizard**.

MODEL WIZARD

1 In the **Model Wizard** window, click **0D**.
2 In the **Select Physics** tree, select **Mathematics>ODE and DAE Interfaces>Global ODEs and DAEs (ge)**.
3 Click **Add**.
4 Click **Study**.
5 In the **Select Study** tree, select **General Studies>Time Dependent**.
6 Click **Done**.

GLOBAL DEFINITIONS

Parameters 1

1 In the **Model Builder** window, under **Global Definitions** click **Parameters 1**.
2 In the **Settings** window for **Parameters**, locate the **Parameters** section.
3 In the table, enter the following settings:

Name	Expression	Value	Description
a	0.25	0.25	ODE constant 1
b	0.05	0.05	ODE constant 2
c	0.015	0.015	ODE constant 3
u0	0.25	0.25	Initial value
f	1[Hz]	1 Hz	Frequency
w	2*pi*f	6.2832 Hz	Angular frequency

GLOBAL ODES AND DAES (GE)

Define the ordinary differential equation with the periodic forcing function.

Global Equations 1

1 In the **Model Builder** window, under **Component 1 (comp1)>Global ODEs and DAEs (ge)** click **Global Equations 1**.
2 In the **Settings** window for **Global Equations**, locate the **Global Equations** section.
3 In the table, enter the following settings:

Name	f(u,ut,utt,t) (1)	Initial value (u_0) (1)	Initial value (u_t0) (1/s)
u	((1/f)*ut-a*sin(w*t)^2+b*u+c*u^2)	u0	0

STUDY 1

The model needs to be solved for 100 periods before it reaches its periodic steady state solution.

Step 1: Time Dependent

1 In the **Model Builder** window, under **Study 1** click **Step 1: Time Dependent**.

2 In the **Settings** window for **Time Dependent**, locate the **Study Settings** section.

3 From the **Tolerance** list, choose **User controlled**.

4 In the **Relative tolerance** text field, type `1e-5`.

5 In the **Output times** text field, type `range(0,0.01,100)`.

Solution 1 (sol1)

1 In the **Study** toolbar, click **Show Default Solver**.

2 In the **Model Builder** window, expand the **Solution 1 (sol1)** node, then click **Time-Dependent Solver 1**.

3 In the **Settings** window for **Time-Dependent Solver**, click to expand the **Absolute Tolerance** section.

4 From the **Tolerance method** list, choose **Manual**.

5 In the **Absolute tolerance** text field, type `0.0001`.

6 In the **Study** toolbar, click **Compute**.

RESULTS

1D Plot Group 1

1 In the **Settings** window for **1D Plot Group**, locate the **Legend** section.

2 From the **Position** list, choose **Lower right**.

3 In the **1D Plot Group 1** toolbar, click **Plot**.

1D Plot Group 2

1 Right-click **Results>1D Plot Group 1** and choose **Duplicate**.

2 In the **Settings** window for **1D Plot Group**, locate the **Axis** section.

3 Select the **Manual axis limits** check box.

4 In the **x minimum** text field, type `90`.

5 In the **y minimum** text field, type `1.6`.

6 In the **y maximum** text field, type `1.7`.

7 In the **1D Plot Group 2** toolbar, click **Plot**.

ROOT

Now add another study with an **Optimization** step which can be used to immediately compute the periodic steady state solution for the differential equation.

ADD STUDY

1 In the **Home** toolbar, click **Add Study** to open the **Add Study** window.

2 Go to the **Add Study** window.

3 Find the **Studies** subsection. In the **Select Study** tree, select **General Studies> Time Dependent**.

4 Click **Add Study** in the window toolbar.

5 In the **Home** toolbar, click **Add Study** to close the **Add Study** window.

STUDY 2

Step 1: Time Dependent

1 In the **Settings** window for **Time Dependent**, locate the **Study Settings** section.

2 In the **Output times** text field, type `range(0,0.002,1)`.

3 From the **Tolerance** list, choose **User controlled**.

4 In the **Relative tolerance** text field, type `1e-5`.

Optimization

1 In the **Study** toolbar, click **Optimization** and choose **Optimization**.

2 In the **Settings** window for **Optimization**, locate the **Optimization Solver** section.

3 From the **Method** list, choose **IPOPT**.

Add the difference between initial and final value in a cycle as error measure to be minimized.

4 Locate the **Objective Function** section. In the table, enter the following settings:

Expression	Description	Evaluate for
`(comp1.u-u0)^2`	`Squared error`	Time Dependent

Next, add the initial value as control parameter and set suitable bounds to help the solver.

5 Locate the **Control Variables and Parameters** section. Click **Add**.

6 In the table, enter the following settings:

Parameter name	Initial value	Scale	Lower bound	Upper bound
u0 (Initial value)	0.25	1	0	5

Solution 2 (sol2)

1 In the **Study** toolbar, click **Show Default Solver**.

2 In the **Model Builder** window, expand the **Solution 2 (sol2)** node.

3 In the **Model Builder** window, expand the **Study 2>Solver Configurations>Solution 2 (sol2)>Optimization Solver 1** node, then click **Time-Dependent Solver 1**.

4 In the **Settings** window for **Time-Dependent Solver**, click to expand the **Absolute Tolerance** section.

5 From the **Tolerance method** list, choose **Manual**.

6 In the **Absolute tolerance** text field, type `1e-5`.

7 In the **Study** toolbar, click **Compute**.

The solver will issue a warning as a reminder that the objective function is only evaluated at the final time — which is indeed the desired behavior in this model.

RESULTS

1D Plot Group 3

1 Click the **Zoom Extents** button in the **Graphics** toolbar.

The periodic steady state solution is obtained (compare to Figure 2).