An analogy between open boundary condition and infinite element domains

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Abstract

A magnetic field associated with a current-carrying coil(s) extends towards infinity, but an analytical model is of finite size. In the finite element analysis (FEA) of electromagnetic systems, open space needs to be modelled as close to reality as possible to obtain reliable results without losing accuracy and by saving computational costs. Using magnetic vector potential (A) formulation, such electromagnetic systems can be modelled by the open boundary condition (OBC), also called the asymptotic boundary condition (ABC) in the literature/FEMM. In COMSOL, there is no explicit ABC, so the current density boundary condition is modified to model the open domain asymptotically. Ampere's law along with the solution of 'A' makes the surface current density boundary condition behave like ABC. In this paper, the FEA problem is set where the self and mutual inductance of two single-turn circular coplanar concentric coils is obtained using OBC and a comparison is made with the infinite element domain of COMSOL and the Dirichlet boundary condition. This paper discusses how the implemented OBC and infinite element domain of COMSOL are analogous and can be used interchangeably. The developed modelling technique can be used in the analytical design of electromagnetic systems giving reliable results within confined design space, saving computational time and resources.

Keywords: Finite element analysis, Open boundary condition, Dirichlet boundary condition, Infinite element domain, FEMM.

I. Introduction

Electromagnetic field designs rely on Finite Element Analysis (FEA) for the accurate prediction of system behaviour before a system is physically built. FEA is a numerical tool that can analyse the electromagnetic system under consideration. The strength of FEA lies in its ability to predict system behaviour accurately or as close to full accuracy as possible. It is essential that the FEA design problem is set as close to reality as possible in order for the system performance factors to be accepted.

Modelling of the magnetic field associated with current-carrying coils in electromagnetic systems is one such design problem. The field extends towards infinity but becomes weaker with increasing distance from its source. Several analytical methods have been devised to model the magnetic field by FEA, such as open boundary condition (OBC). OBC utilizes magnetic vector potential (A) formulation and can be implemented in FEA packages like Finite Element Methods Magnetics (FEMM) and COMSOL Multiphysics.

The FEMM electromagnetic simulator implements OBC by naming it as an asymptotic boundary condition (ABC) [3]. COMSOL is a widely accepted and used FEA software for engineering designs but does not have explicit ABC. However, in COMSOL other boundary conditions can be made to behave like ABC [2]. COMSOL has devised a way of modelling an open-space magnetic field in the infinite element domain.

In this paper, the COMSOL infinite element domain is shown to be analogous to ABC. The FEA problem is set up, and self and mutual inductance between two coplanar circular single-turn coils is obtained in AC and DC settings. The magnetic field associated with the open space is modelled by magnetic vector potential (A) formulation. The theory of the magnetic field and its characteristics in open space are arranged in the form of a boundary condition. When this boundary surrounds a design space, it gives the effect of a magnetic field in open space. The developed boundary condition is implemented in COMSOL, and the results are compared with ABC in FEMM, as well as the infinite element domain in COMSOL.

Section II of the paper details the theory related to the open boundary condition in FEMM, and OBC is implemented in COMSOL by utilizing and modifying the surface current density boundary condition. In section III, the FEA problem is set up and the coplanar single-turn circular coils are modelled. Section IV gives an analytic solution of self and mutual inductance in the problem setup. Section V provides a detailed comparison of the results through ABC in FEMM & COMSOL and the infinite element domain in COMSOL. The developed problem is analysed in steady-state and frequency domain setups. The advantage of ABC is highlighted where it is claimed to be useful in reducing computation time and model size compared to the Dirichlet boundary condition. Section VI concludes the paper.

II. Open Boundary Condition

In electromagnetic problems, FEMM approximates open space by ABC (in this paper the two abbreviations ABC and OBC are used interchangeably). The problem domain is a circular shell of radius r in an unbounded region. As the radius of the domain approaches infinity, the magnetic vector potential (A) approaches zero. On the surface of the circle, 'A' is a prescribed function of θ and is given by:

$$\mathrm{A}(\mathrm{r},\theta) = \sum_{m=1}^{\infty} \frac{\mathrm{a_m}}{\mathrm{r^m}} \cos(\mathrm{m}\theta + \alpha_\mathrm{m}) \tag{1}$$

'A' is a harmonic function and the expansion of the above expression shows that increasing the number of harmonics leads to the faster decay of their magnitude with increasing r. This is exactly correct at infinity but only approximately correct when imposed at a finite boundary. With n as the number of leading harmonics, the relation between 'A' and its derivative on a circular artificial boundary of radius r is:

$$\frac{\partial \mathrm{A}}{\partial \mathrm{r}} + (\frac{\mathrm{n}}{\mathrm{r}})\mathrm{A} = 0 \tag{2}$$

which is recognised as an asymptotic boundary condition (ABC) and FEMM supports this as a "mixed" boundary condition with the form:

$$\frac{1}{\mu_\mathrm{o}\mu_\mathrm{r}}(\frac{\partial \mathrm{A}}{\partial \mathrm{r}}) + \mathrm{c_o}\mathrm{A} + \mathrm{c_1} = 0$$

where

$$c_o = \frac{n}{\mu_o \mu_r r}$$

$$c_1 = 0$$

COMSOL does not have such boundary conditions, so one of its built-in boundaries, the surface current density boundary condition, is modified to model the open domain asymptotically. The surface current density boundary condition yeilds

$$-n \times H = J_s$$

J_s is the surface current density. It can be obtained from Ampere's law and using the relation between magnetic field intensity (H) and magnetic flux density (B).

$$J = \nabla \times H = \nabla \times (\frac{B}{\mu}) = \nabla \times (\frac{\nabla \times A}{\mu})$$

using eq. (2)

$$J = \frac{\nabla \times (-r^{-1}A)}{\mu} = -\frac{\nabla \times A}{r\mu} = -\frac{A_{phi}}{r\mu} \quad (3)$$

(A_{phi} is the phi component of Magnetic vector potential). Equating eq. (3) and Ampere's law causes the surface current density boundary condition to behave like a 1^{st} order ABC. In COMSOL 2D axisymmetric model, this is achieved (Figure 1) by implementing the expression

$$J_{so} = -\frac{A_{phi}}{mu0\_const * sqrt(r*r + z*z)}$$

which is by default set to zero.

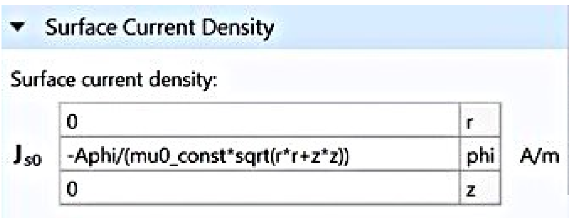


Figure 1

III: Problem geometry setup

The geometrical arrangement of the two single-turn circular coils in a concentric arrangement is shown in Figure 2. If these coils are arranged in a concentric planar form with condition $R_1 \gg R_2 \gg r_o$, the resulting geometry becomes that which is shown in Figure 3. R_1 is the radius of the outer coil. R_2 is the radius of the inner coil and r_o is the radius of both conductors. Under the condition $R_1 \gg R_2 \gg r_o$ the problem geometry becomes that which is shown in Figure 4. The coil 1 carries the current, while the coil 2 is open-circuited. The self-inductance of a current-carrying coil (L_{11}) and the mutual inductance (M_{12}) between the two coils are computed to make analogies and comparisons.

IV: Analytical solution

The analytic solution for self and mutual inductance can be obtained for this problem. If a current of 1 Amp is switched on in coil 1 (Figure 4), the self-inductance of coil 1 in free space is given by [4]

$$L_{11} = \mu_o R_1 [\ln(\frac{8R_1}{r_o}) - 1.75] \quad (4)$$

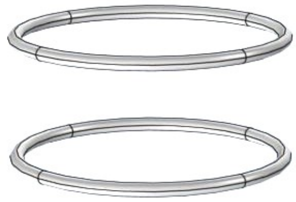


Figure 2: Concentric single-turn coils

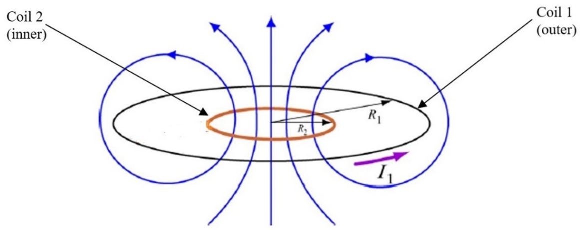


Figure 3: Concentric coplanar single-turn coils

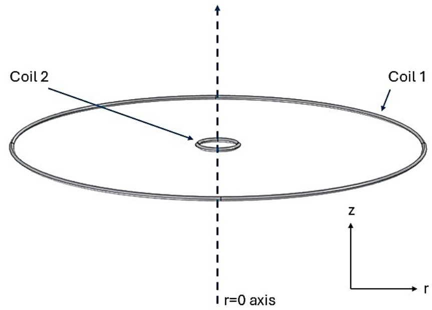


Figure 4: Resulting geometry

The mutual inductance for coil 2 is obtained using the Biot-Savart law (Figure 5).

The differential element of magnetic field (dB) due to the elemental length (dl) is

$$dB = (\frac{\mu_o I}{4\pi})(\frac{dl\ \sin 90^0}{r^2})$$

All $dB\cos\theta$ cancel out giving a total magnetic field density of

$$B = \sum dB\ \sin\theta$$

$$B = (\frac{\mu_o}{4\pi}) \oint (\frac{dl\ \sin\theta}{r^2})$$

$$B = \frac{\mu_o R_1^2}{2(R_1^2 + x^2)^(3/2)}$$

At the axis, x=0

$$B = \frac{\mu_o}{2(R_1)}$$

Mutual inductance for the inner coil is given as

$$M_{12} = B(\text{area of coil 2})$$

$$M_{12} = (\frac{\mu_o}{2(R_1)})(\pi R_2^2) \quad (5)$$

The analytic solutions in eqs. (4) and (5) are valid under the assumption $R_1 \gg R_2 \gg r_o$; otherwise, the classical solution involves elliptic integrals [5].

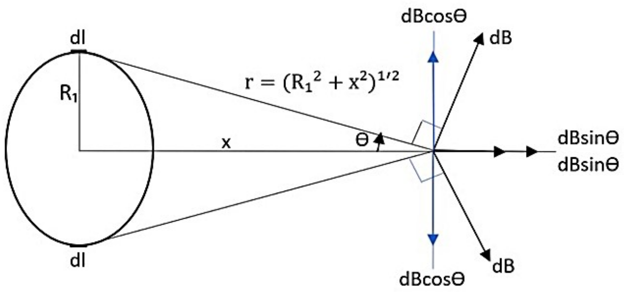


Figure 5: Biot savart law

Parameters
radius of coil 1=R_1=100mm
radius of coil 2=R_2=10mm
radius of wire=r_0=1mm
current in coil 1=1Amp
current in coil 2=0 (open circuited)

With the above parameters, the self-inductance for coil 1 and mutual inductance yields the analytic values:

$$L_{11} = 6.19208 \times 10^{-7}\mathrm{H} \tag{6}$$

$$M_{12} = 1.97192 \times 10^{-9}\mathrm{H} \tag{7}$$

These values are used to make the comparisons presented in the next section.

Section V: Comparison between Open boundary condition and Infinite element domain

A comparison is first made under a steady-state (DC) condition and later in the frequency domain (AC).

DC Analysis

2D model forms

A 2D axisymmetric problem is set up where the quantities are independent of the azimuthal component. Models in two solvers are:

- FEMM with asymptotic boundary condition
- COMSOL with asymptotic boundary condition
- COMSOL with infinite element domain

Results from the above models are also compared with 3D equivalent geometry in COMSOL using an infinite element domain.

FEMM with asymptotic boundary condition

Using defined parameters, the FEMM 2D axisymmetric problem is set up. The outer boundary is modelled with ABC. At varying numbers of harmonics, n=1, 2, 3, 5, 10, and 20, plots of magnetic flux density and magnetic vector potential (from coil 1 to the outer boundary) are obtained. For n=1 and n=20, these are given in Figures 6 and 7.

The results show that as the number of harmonics (significant terms in eq. (1)) increases while keeping the boundary at a constant distance, more flux permeated the domain. However, an error in self and mutual inductance values increases with an increasing number of harmonics,

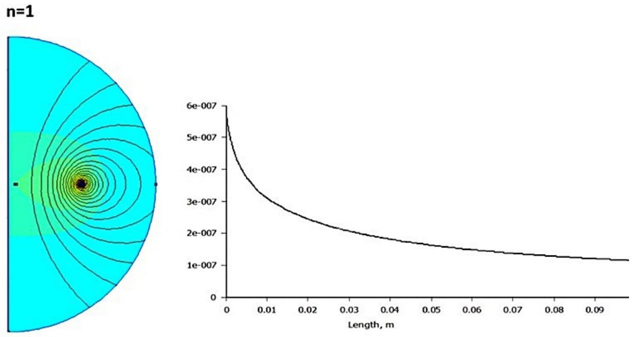


Figure 6: 1^{st} order ABC, Magnetic flux density (left), Magnetic potential from coil 1 to outer boundary (right)

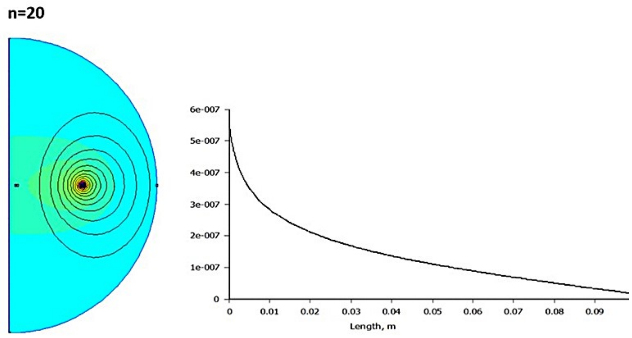


Figure 7: 20^{th} order ABC, Magnetic flux density (left), Magnetic potential from coil 1 to outer boundary (right)

as shown in the Table 1. This is because as the number of harmonics increases, the potential at the boundary increases (second term in eq. (2)) leading to an increasing error value.

n	L_{11} (10^{-7}H)	Error L_{11} (%)	M_{12} (10^{-9}H)	Error M_{12} (%)
1	6.115	1.25	1.975	0.17
2	6.052	2.27	1.914	2.95
3	6.014	2.88	1.877	4.82
5	5.97	3.59	1.835	6.96
10	5.924	4.33	1.791	9.19
20	5.895	4.80	1.763	10.61

Table 1: Self & mutual inductance at varying numbers of harmonics in ABC

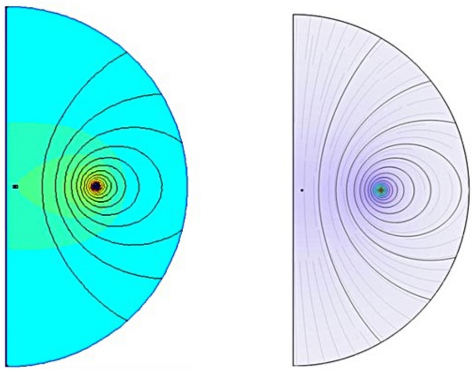
COMSOL with asymptotic boundary condition

Using defined parameters, the COMSOL 2D axisymmetric problem is set up. The outer boundary is modelled with the surface current density boundary condition in the form of ABC (Figure 1). The obtained self and mutual inductance values are:

$$L_{11} = 6.2033 \times 10^{-7}\mathrm{H}$$

$$M_{12} = 1.9764 \times 10^{-9}\mathrm{H}$$

Both of these values have an error of 0.2% when compared to the analytically calculated values in eqs. (6) and (7). The magnetic flux density obtained in FEMM and COMSOL for 1^{st} order ABC under given conditions is shown in Figure 8. The results illustrate that there is a good comparison between



(a) *(b)*

Figure 8: Magnetic flux density with 1^{st} *order ABC (a) FEMM (b) COMSOL*

the results obtained from both solvers and ABC is correctly implemented in COMSOL.

COMSOL with infinite element domain

COMSOL has a built-in feature, the infinite element domain, to model open boundaries. This infinite element domain applies a semi-infinite coordinate stretching in one, two, or three directions, depending on how the infinite element domain connects to the physical domain. The COMSOL 2D axisymmetric model was tested with an infinite element domain at the outer boundary and magnetic flux density was found to be the same as in Figure 8(b). Self and mutual inductance were found to be 6.1938×10^{-7}H and 1.9746×10^{-9}H which depicts an error of 0.02% and 0.1% when compared to the analytical values in eqs. (6) and (7).

COMSOL 3D Implementation with infinite element domain

The 3D version of the problem is implemented in COMSOL as shown in Figure 9.

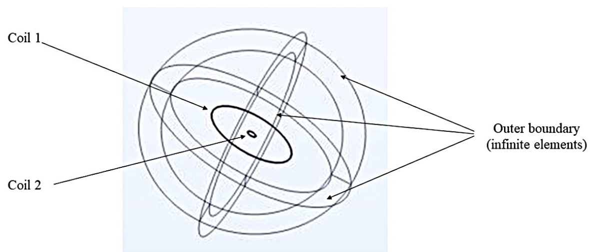


Figure 9: 3D Topology

The self and mutual inductance obtained from this model were:

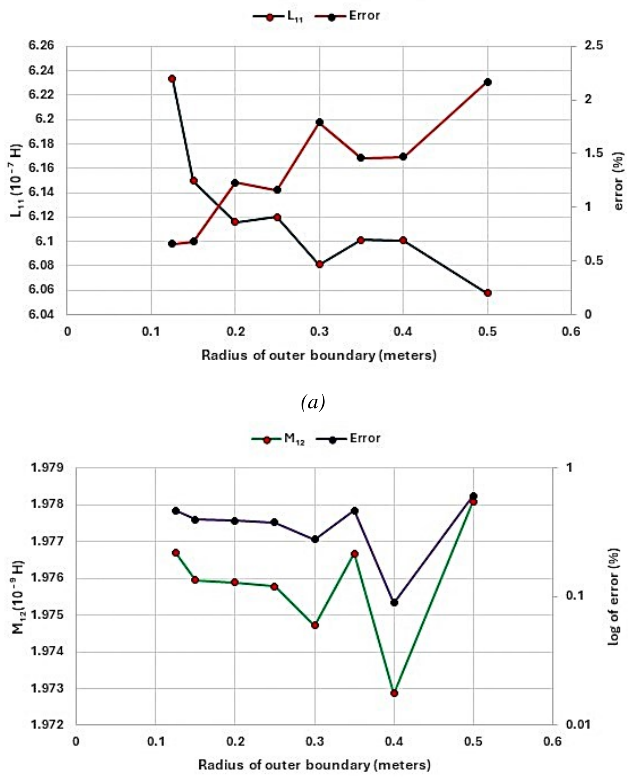
$$L_{11} = 6.192 \times 10^{-7}\text{H}$$

$$M_{12} = 1.975 \times 10^{-9}\text{H}$$

demonstrating a notable similarity to the analytically calculated values in eqs. (6) and (7). This shows a 3D COMSOL model of a given problem, with an infinite element domain, compares well with analytic 2D FEMM (ABC), 2D COMSOL (ABC), and 2D COMSOL (infinite element domain) solutions.

Advantage of ABC

The 2D model is used to make a comparison between 1^{st} order ABC and A=0 (Dirichlet (FEMM) /Magnetic insulation(COMSOL)) conditions. Self and mutual inductance are obtained while varying the radius of the outer boundary, increasing its value from 0.15m to 0.5m. The self and mutual inductance at each radius value and the percentage error from their respective analytic values (see eq. (5) and (6)) are shown in graphs of Figure 10.



(a)

(b)

Figure 10: (a) Self inductance VS Error (1^{st} *order ABC (b) Mutual inductance VS Error (*1^{st} *order ABC)*

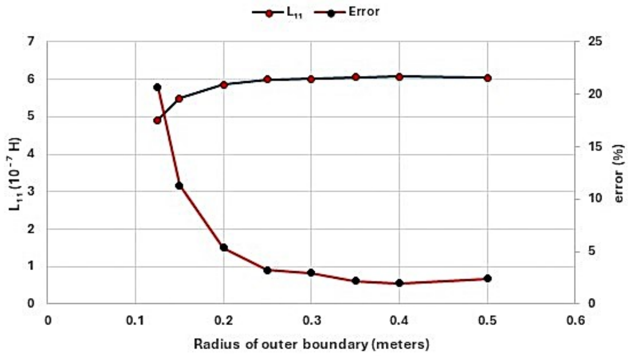
With the increasing radius of the outer boundary, 1^{st} order ABC shows better results as an error in self and mutual inductance is a maximum of 2.17% and 0.61% respectively (Figure 10), which is reasonable.

With A=0, Dirichlet/Magnetic insulation boundary the error is a maximum when the boundary is closer to the working domain, that is 20.7% for L_{11} and 50.9% for M_{12} (Figure 11). The error decreases as the radius of the boundary is increased.

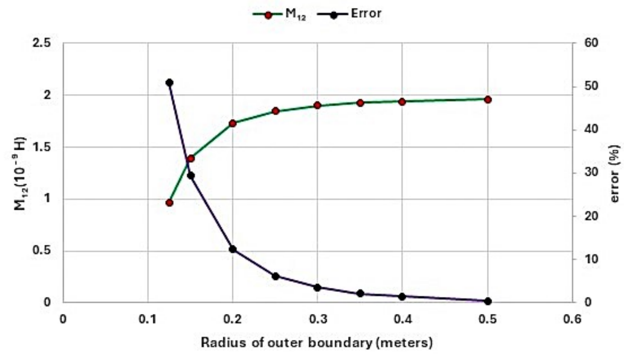
It can be said that ABC, when models open boundary, can be placed closer to the working domain and one can still get close to accurate results; however, the A=0 boundary condition needs to be placed at the outer boundary far away from the working domain, typically at five times or higher the radius of the working domain. So, ABC can reduce the size of the model and save computation time.

AC Analysis

The 2D FEMM ABC model, 2D axisymmetric COMSOL model, and 3D infinite element domain model (developed in the last section) are studied in the frequency domain (AC). A current of 1 Amp at 1kHz is turned on in the coil 1. The induced voltage in coil 2 (modelled as an open circuit) and mutual inductance are obtained. In the time-harmonic



(a)



(b)

Figure 11: (a) Self-inductance VS Error (Dirichlet/Magnetic insulation) (b) Mutual inductance VS Error ((Dirichlet/Magnetic insulation)

(frequency domain) case, the mutual inductance is computed as:

$$\mathrm{M}_{12} = \frac{\mathrm{V}_2}{i\omega \mathrm{I}_1} \tag{8}$$

where V_2 is the induced voltage in coil 2 and I_1 is the current in coil 1. The induced voltage and mutual inductance found with developed models are given in Table 2. Values

Solver	**Boundary condition**	**Model**	$\mathbf{V}_2$ $(10^{-5}\mathrm{V})$	$\mathbf{L}_{12}$ $(10^{-9}\mathrm{H})$
FEMM	ABC	2D	0.00231 +1.24162i	1.9761 -0.00368i
COMSOL	ABC	2D	0.00240 +1.2416i	1.9761 -0.00382i
COMSOL	infinite elements	2D	0.00240 +1.2412i	1.9754 -0.00382i
COMSOL	infinite elements	3D	0.00238 +1.2416i	1.9760 -0.00378i

Table 2: Induced voltage and mutual inductance under AC conditions

from all three models resemble each other closely. This further verifies that ABC and the infinite element domain give the same results and are analogous. The computed mutual inductance has a small imaginary component and this is due to the resistive effects. There are eddy current losses in the wires, due to finite conductivity, and the coil AC impedance, though mainly reactive, has a small resistive part [1].

Section VI: Conclusion

The COMSOL surface current boundary condition is modified to model an open-space magnetic field asymptotically. This boundary is compared to FEMM ABC by measuring the self-inductance and mutual inductance between two coplanar concentric single-turn circular coils. The outer boundary in the COMSOL model is replaced by an infinite element domain and the results are found to be the same as those obtained for the ABC model. In an analysis of AC and DC conditions, ABC and infinite element domain models give the same values of self and mutual inductance related to the developed FEA problem. At the outer boundary of FEA design space, the ABC and infinite element domain are found to model an open-space magnetic field. Thus, the ABC and infinite element domain are analogous and can be used in place of each other.

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